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Analysis of Variance of Partially Balanced Fractional

 $2^{m_1+m_2}$ Factorial Designs of Resolution IV

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Abstract

In this paper, attention is focused on the analysis of variance of partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution IV by using the algebraic structure. They can be obtained by partially balanced arrays with some conditions.

1. Introduction

A partially balanced array (PB-array), which is a special case of an asymmetrical balanced array of type 2 as introduced by Nishii [14], has been studied by several researchers (e.g., [4]). Necessary and sufficient conditions for the existence of a PB-array were obtained by Kuwada and Kuriki [10]. A PB-array yields a partially balanced fractional $2^{m_1+m_2}$ factorial ($2^{m_1+m_2}$ -PBFF) design under some conditions (see [5,6]). However a $2^{m_1+m_2}$ -PBFF design does not always mean a PB-array.

It is generally difficult to obtain the designs of resolution 2ℓ since there is a little information about the ℓ -factor interactions. For earlier works on such designs, see for example, Kuwada and/or Matsuura [3,11], Margolin [12,13], Shirakura [17-20], Srivastava and/or Anderson [1,22], and Webb [23]. Especially, by using the triangular multidimensional partially bal-

anced (TMDPB) association scheme and its algebra, Shirakura [17] showed that a balanced array with index $\mu_0=0$ turns out to be a balanced fractional 2^m factorial design of resolution 2ℓ under some conditions. Such a design permits to estimate all factorial effects up to the $(\ell-1)$ -factor interactions and some linear combinations of the ℓ -factor ones.

The analysis of variance (ANOVA) is a statistical technique for handling the data or observations derived from an experiment (cf. [9,15,16]). The ANOVA of $2^{m_1+m_2}$ -PBFF designs of resolution V which are derived from PB-arrays has been studied by Kuwada [8]. In this paper, we present the ANOVA and the hypothesis testing of $2^{m_1+m_2}$ -PBFF designs of resolution IV, which are PB-arrays. The designs considered here permit estimation of the general mean, all main effects and (A) all $\binom{m_1}{2} + \binom{m_2}{2}$ two-factor interactions and some linear combinations of the $m_1 m_2$ ones, (B) all $\binom{m_1}{2}$ ones and some linear combinations of the $\binom{m_2}{2}$ ones and of the $m_1 m_2$ ones, or (C) some linear combinations of the $\binom{m_k}{2}$ ones ($k=1,2$) and of the $m_1 m_2$ ones (see [3,11]).

2. Preliminaries

Consider a factorial experiment with m_1+m_2 factors at two levels (0 and 1, say) of each, where $m_k \geq 2$. Further consider the situation in which three-factor and higher order interactions are assumed to be negligible. The vector of unknown factorial effects to be estimated is then given by $(\theta'_{00}; \theta'_{10}; \theta'_{01}; \theta'_{20}; \theta'_{02}; \theta'_{11}) (= \theta', \text{ say})$, where $\theta'_{00} = (\{\theta(0;0)\})$, $\theta'_{10} = (\{\theta(u;0)\})$, $\theta'_{01} = (\{\theta(0;v)\})$, $\theta'_{20} = (\{\theta(u_1 u_2; 0)\})$, $\theta'_{02} = (\{\theta(0; v_1 v_2)\})$ and $\theta'_{11} = (\{\theta(u; v)\})$. Here $1 \leq u \leq m_1$, $1 \leq v \leq m_2$, $1 \leq u_1 < u_2 \leq m_1$ and $1 \leq v_1 < v_2 \leq m_2$, and A' denotes the

transpose of a matrix A . Note that the total number of factorial effects to be estimated is $1+(m_1+m_2)+\binom{m_1+m_2}{2}$ ($=\nu(m_1m_2)$, say). Let $[T^{(1)};T^{(2)}](=T, \text{ say})$ be a fraction with N assemblies (or treatment combinations), where $T^{(k)}$'s are $(0,1)$ -matrices of size $N \times m_k$. Then the ordinary linear model is given by

$$y_T = E_T \theta + e_T, \quad (2.1)$$

where y_T and E_T are the vector of N observations and the design matrix of size $N \times \nu(m_1m_2)$, respectively, and e_T is an error vector distributed as $N(0_N, \sigma^2 I_N)$. Here 0_p and I_p denote the $p \times 1$ vector with all zero and the identity matrix of order p , respectively. The normal equation for estimating θ is given by $M_T \hat{\theta} = E_T' y_T$, where $M_T = E_T' E_T$. If the information matrix M_T is nonsingular, the BLUE of θ and its variance-covariance matrix are given by $\hat{\theta} = M_T^{-1} E_T' y_T$ and $\text{Var}[\hat{\theta}] = \sigma^2 M_T^{-1}$, respectively.

Suppose a relation of association is defined among the sets $\{(u_1 \cdots u_{a_1}; v_1 \cdots v_{a_2})\}$, where $1 \leq u_1 < \cdots < u_{a_1} \leq m_1$ and $1 \leq v_1 < \cdots < v_{a_2} \leq m_2$, in such a way that $(u_1 \cdots u_{a_1}; v_1 \cdots v_{a_2})$ and $(u'_1 \cdots u'_{b_1}; v'_1 \cdots v'_{b_2})$ are the $(\alpha_1 \alpha_2)$ th associates if

$$|\{u_1, \dots, u_{a_1}\} \cap \{u'_1, \dots, u'_{b_1}\}| = \min(a_1, b_1) - \alpha_1$$

and

$$|\{v_1, \dots, v_{a_2}\} \cap \{v'_1, \dots, v'_{b_2}\}| = \min(a_2, b_2) - \alpha_2,$$

where $|S|$ and $\min(a, b)$ denote the cardinality of a set S and the minimum value of integers a and b , respectively. The scheme thus defined is called the extended TMDPB (ETMDPB) association scheme (see [5]), which is regarded as a generalization of the TMDPB association scheme (e.g., [24, 25]). Let $A_{\alpha_1 \alpha_2}^{(a_1 a_2, b_1 b_2)}$ and $D_{\alpha_1 \alpha_2}^{(a_1 a_2, b_1 b_2)}$ be the local association matrices of size $n(a_1 a_2) \times$

$n(b_1 b_2)$ and the ordered association matrices of order $\nu(m_1 m_2)$ of the ETMDPB association scheme, respectively (see [5]), where $n(a_1 a_2) = \binom{m_1}{a_1} \binom{m_2}{a_2}$. Further let $A_{\beta_1 \beta_2}^{\#(a_1 a_2, b_1 b_2)} = A_{\beta_1}^{\#(a_1, b_1)} \otimes A_{\beta_2}^{\#(a_2, b_2)}$, where $A_{\beta}^{\#(a, b)}$'s are the matrices which are linearly linked with the local association matrices $A_{\alpha}^{(a, b)}$ of the TMDPB association scheme (e.g., [25]), and \otimes denotes the Kronecker product. A relationship between $A_{\alpha}^{(a, b)}$'s and $A_{\beta}^{\#(a, b)}$'s is given by

$$A_{\alpha}^{(a, b)} = \{A_{\alpha}^{(b, a)}\}' = \sum_{\beta} Z_{\beta \alpha}^{(a, b)} A_{\beta}^{\#(a, b)} \quad \text{for } 0 \leq a \leq b \leq m$$

and

$$A_{\beta}^{\#(a, b)} = \{A_{\beta}^{\#(b, a)}\}' = \sum_{\alpha} Z_{\alpha \beta}^{(a, b)} A_{\alpha}^{(a, b)} \quad \text{for } 0 \leq a \leq b \leq m,$$

where

$$Z_{\beta \alpha}^{(a, b)} = \sum_{p=0}^{\alpha} (-1)^{p-\alpha} \binom{a-p}{p} \binom{a-p}{\alpha-p} \binom{m-a-p}{p} \{ \binom{m-b-a-p}{p} \binom{b-p}{a-p} \}^{1/2} / \binom{b-a+p}{p}$$

for $a \leq b$,

$$Z_{\alpha \beta}^{(a, b)} = \phi_{\beta} Z_{\beta \alpha}^{(a, b)} / \{ \binom{m}{\alpha} \binom{a}{\alpha} \binom{m-a}{b-a+\alpha} \} \quad \text{for } a \leq b$$

and

$$\phi_{\beta} = \binom{m}{\beta} - \binom{m}{\beta-1}$$

(e.g., [7, 21, 25]). The matrices $A_{\beta_1 \beta_2}^{\#(a_1 a_2, b_1 b_2)}$ have the following

properties:

$$A_{00}^{\#(a_1 a_2, b_1 b_2)} = [1/\{n(a_1 a_2) \times n(b_1 b_2)\}^{1/2}] G_{n(a_1 a_2) \times n(b_1 b_2)}, \quad (2.2)$$

$$\sum_{\beta_1 \beta_2} A_{\beta_1 \beta_2}^{\#(a_1 a_2, a_1 a_2)} = I_{n(a_1 a_2)}, \quad (2.3)$$

$$A_{\beta_1 \beta_2}^{\#(a_1 a_2, c_1 c_2)} A_{\gamma_1 \gamma_2}^{\#(c_1 c_2, b_1 b_2)} = \delta_{\beta_1 \gamma_1} \delta_{\beta_2 \gamma_2} A_{\beta_1 \beta_2}^{\#(a_1 a_2, b_1 b_2)} \quad (2.4)$$

and

$$\text{rank}(A_{\beta_1 \beta_2}^{\#(a_1 a_2, b_1 b_2)}) = \phi_{\beta_1} \times \phi_{\beta_2} (= \phi_{\beta_1 \beta_2}, \text{ say}) \quad (2.5)$$

(see [5]), where $G_{p \times q}$ and δ_{pq} denote the $p \times q$ matrix with all unity and the Kronecker delta, respectively.

Let T be a PB-array of strength $t_1 + t_2$ and size N having $m_1 + m_2$ constraints, two levels, and index set $\{\mu(i_1 i_2) \mid 0 \leq i_k \leq t_k \leq m_k\}$,

where

$$t_k = \begin{cases} m_k & \text{if } m_k = 2, 3, \\ 4 & \text{if } m_k \geq 4. \end{cases} \quad (2.6)$$

This array is written as $PBA(N, m_1 + m_2, 2, t_1 + t_2; \{\mu(i_1 i_2)\})$ for brevity. The information matrix M_T associated with T , which is a PB-array with (2.6), can be expressed as

$$\begin{aligned} M_T &= \sum_{a_1 a_2} \sum_{b_1 b_2} \sum_{\alpha_1 \alpha_2} \gamma_{|a_1 - b_1| + 2\alpha_1, |a_2 - b_2| + 2\alpha_2} D_{\alpha_1 \alpha_2}^{(a_1 a_2, b_1 b_2)} \\ &= \sum_{a_1 a_2} \sum_{b_1 b_2} \sum_{\beta_1 \beta_2} \kappa_{\beta_1 \beta_2}^{a_1 a_2, b_1 b_2} D_{\beta_1 \beta_2}^{\#(a_1 a_2, b_1 b_2)}, \end{aligned} \quad (2.7)$$

where $D_{\beta_1 \beta_2}^{\#(a_1 a_2, b_1 b_2)}$'s are the matrices of order $\nu(m_1 m_2)$ which are

given by some linear combinations of $D_{\alpha_1 \alpha_2}^{(a_1 a_2, b_1 b_2)}$'s,

$$\gamma_{j_1, j_2} = \sum_{i_1=0}^{t_1} \sum_{i_2=0}^{t_2} \left[\prod_{k=1}^2 \left\{ \sum_{p_k=0}^{j_k} (-1)^{p_k} \binom{j_k}{p_k} \binom{t_k - j_k}{i_k - j_k + p_k} \right\} \right] \mu(i_1 i_2)$$

and

$$\kappa_{\beta_1 \beta_2}^{a_1 a_2, b_1 b_2} = \sum_{\alpha_1 \alpha_2} \left[\prod_{k=1}^2 \left\{ z_{\beta_k \alpha_k}^{(a_k, b_k)} \right\} \right] \gamma_{|a_1 - b_1| + 2\alpha_1, |a_2 - b_2| + 2\alpha_2}$$

(see [5]).

3. $2^{m_1 + m_2}$ -PBFF designs of resolution IV

Throughout this paper, we consider a design, which is a PBA $(N, m_1 + m_2, 2, t_1 + t_2; \{\mu(i_1 i_2)\})$ with (2.6). Let $K_{\beta_1 \beta_2} = \|\kappa_{\beta_1 \beta_2}^{a_1 a_2, b_1 b_2}\|$ for $\beta_1 \beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11 . Then a necessary and sufficient condition for the information matrix M_T to be nonsingular, i.e., T is of resolution V, is that every $K_{\beta_1 \beta_2}$ is positive definite (see [5]). Note that a PBA $(N, m_1 + m_2, 2, t_1 + t_2; \{\mu(i_1 i_2)\})$ yields a $2^{m_1 + m_2}$ -PBFF design of resolution V provided M_T is nonsingular. However the converse is not always true.

In this paper, we consider three cases as follows:

(A) $\det(K_{\beta_1 \beta_2}) \neq 0$ for $\beta_1 \beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$),

and $\det(K_{11})=0$,

(B) $\det(K_{\beta_1\beta_2}) \neq 0$ for $\beta_1\beta_2=00,10,01,20$ (if $m_1 \geq 4$), and $\det(K_{02}) = \det(K_{11})=0$ for $m_2 \geq 4$

and

(C) $\det(K_{\beta_1\beta_2}) \neq 0$ for $\beta_1\beta_2=00,10,01$, and $\det(K_{20})=\det(K_{02}) = \det(K_{11})=0$ for $m_k \geq 4$

(see [3,11]), where $\det(A)$ denotes the determinant of a matrix A .

Let $\Psi_A=(\theta'_{00};\theta'_{10};\theta'_{01};\theta'_{20};\theta'_{02};(H^A_{11}\theta_{11})')$, $\Psi_B=(\theta'_{00};\theta'_{10};\theta'_{01};\theta'_{20};(H^B_{02} \times \theta_{02})';(H^B_{11}\theta_{11})')$ for $m_2 \geq 4$ and $\Psi_C=(\theta'_{00};\theta'_{10};\theta'_{01};(H^C_{20}\theta_{20})';(H^C_{02} \times \theta_{02})';(H^C_{11}\theta_{11})')$ for $m_k \geq 4$, where

$$H^A_{11} = h^A_{00}A^*_{00}(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) + h^A_{10}A^*_{10}(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) + h^A_{01}A^*_{01}(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}),$$

$$H^B_{11} = h^B_{00}A^*_{00}(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) + h^B_{10}A^*_{10}(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) + h^B_{01}A^*_{01}(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}),$$

$$H^B_{02} = h^B_{00}A^*_{00}(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}) + h^B_{01}A^*_{01}(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}),$$

$$H^C_{11} = h^C_{00}A^*_{00}(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) + h^C_{10}A^*_{10}(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) + h^C_{01}A^*_{01}(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}),$$

$$H^C_{02} = h^C_{00}A^*_{00}(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}) + h^C_{01}A^*_{01}(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}),$$

$$H^C_{20} = h^C_{00}A^*_{00}(\begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix}) + h^C_{10}A^*_{10}(\begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix}),$$

and $h^A_{\beta_1\beta_2}$'s, $h^{Ba_1a_2}_{\beta_1\beta_2}$'s and $h^{Ca_1a_2}_{\beta_1\beta_2}$'s are real constants. Then we have the following (see [3,11]):

Proposition 3.1. *Let T be a design which satisfies Condition (A) ((B) or (C)). Then Ψ_A (Ψ_B or Ψ_C) is an estimable function of θ , and the BLUE of Ψ_A (Ψ_B or Ψ_C) is given by $\hat{\Psi}_A=X_A E'_{TY_T}$ ($\hat{\Psi}_B=X_B E'_{TY_T}$ or $\hat{\Psi}_C=X_C E'_{TY_T}$), where X_A (X_B or X_C) is a matrix of order $\nu(m_1m_2)$ which satisfies $X_A M_T=Z_A$ ($X_B M_T=Z_B$ or $X_C M_T=Z_C$), $Z_A=\text{diag}(I_{\nu_A}; H^A_{11})$ ($Z_B=\text{diag}(I_{\nu_B}; H^B_{02}; H^B_{11})$ or $Z_C=\text{diag}(I_{\nu_C}; H^C_{20}; H^C_{02}; H^C_{11})$) and $\nu_A=1+m_1+m_2 + (\frac{m_1}{2}) + (\frac{m_2}{2})$ ($\nu_B=1+m_1+m_2 + (\frac{m_1}{2})$ or $\nu_C=1+m_1+m_2$).*

Note that a design satisfying Condition (A) ((B) or (C)) is of course of resolution IV.

4. Algebraic structure

It is empirically known that the main effects are more important than the two-factor interactions. Thus we are interested in testing the hypotheses such that there exist some linear combinations of the two-factor interactions or not. If they do not exist, we wish to test the hypotheses such that there exist another linear combinations of them (or some linear combinations of the main effects) or not, and so on.

Using the properties of $A_{\beta_1\beta_2}^{\#(a_1a_2, a_1a_2)}$,_s as in (2.3), the linear model (2.1) can be rewritten as

$$y_T = \sum_{\beta_1\beta_2} \sum_{a_1a_2} E_{a_1a_2} A_{\beta_1\beta_2}^{\#(a_1a_2, a_1a_2)} \theta_{a_1a_2} + e_T,$$

where $E_{a_1a_2}$'s are $N \times n(a_1a_2)$ submatrices of E_T corresponding to $\theta_{a_1a_2}$, i.e., $E_T = [E_{00}; E_{10}; E_{01}; E_{20}; E_{02}; E_{11}]$. By (2.2), (2.4) and (2.5), (i) every element of the vector $A_{00}^{\#(a_1a_2, a_1a_2)} \theta_{a_1a_2}$ represents the average of $\theta_{a_1a_2}$ for $a_1a_2=00, 10, 01, 20, 02, 11$, (ii) the elements of $A_{\beta_1\beta_2}^{\#(a_1a_2, a_1a_2)} \theta_{a_1a_2}$ for $\beta_1\beta_2=10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11 represent the contrasts between these effects and any two contrasts are orthogonal, and (iii) there exist $\phi_{\beta_1\beta_2}$ independent parametric functions of $\theta_{a_1a_2}$ in $A_{\beta_1\beta_2}^{\#(a_1a_2, a_1a_2)} \theta_{a_1a_2}$, respectively (e.g., [17]).

Let $F_{\beta_1\beta_2}^{a_1a_2, b_1b_2} = E_{a_1a_2} A_{\beta_1\beta_2}^{\#(a_1a_2, b_1b_2)} E_{b_1b_2}'$. Then by (2.4) and (2.7), we get the following (see [8]):

Lemma 4.1.

$$F_{\beta_1\beta_2}^{a_1a_2, c_1c_2} F_{\gamma_1\gamma_2}^{d_1d_2, b_1b_2} = \delta_{\beta_1\gamma_1} \delta_{\beta_2\gamma_2} \kappa_{\beta_1\beta_2}^{c_1c_2, d_1d_2} F_{\beta_1\beta_2}^{a_1a_2, b_1b_2}.$$

Let $K_{\beta_1\beta_2}(a_1a_2)$ be the matrices which are composed of the

initial, ..., the $a_1 a_2$ th rows and the initial, ..., the $a_1 a_2$ th columns of $K_{\beta_1 \beta_2}$. Further let $K_{\beta_1 \beta_2}(a_1 a_2)^{-1} = \|\eta_{\beta_1 \beta_2}^{c_1 c_2, d_1 d_2}(a_1 a_2)\|$, if $K_{\beta_1 \beta_2}(a_1 a_2)$ is nonsingular. In addition, let $K_{\beta_1 \beta_2}(a_1 a_2^*)^{-1} = \|\eta_{\beta_1 \beta_2}^{e_1 e_2, f_1 f_2}(a_1 a_2^*)\|$, if $K_{\beta_1 \beta_2}(a_1 a_2^*)$ is nonsingular, where $K_{\beta_1 \beta_2}(a_1 a_2^*)$ is the matrix which is obtained by deleting the last row and the last column of $K_{\beta_1 \beta_2}(a_1 a_2)$, and $\eta_{\beta_1 \beta_2}^{\beta_1 \beta_2, \beta_1 \beta_2}(\beta_1 \beta_2^*) = 0$ for $\beta_1 \beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11 .

Let

$$P_{\beta_1 \beta_2}^{a_1 a_2} = \frac{a_1 a_2}{\sum_{\beta_1 \beta_2}^*} \frac{a_1 a_2}{\sum_{\beta_1 \beta_2}^*} \eta_{\beta_1 \beta_2}^{c_1 c_2, d_1 d_2}(a_1 a_2) F_{\beta_1 \beta_2}^{c_1 c_2, d_1 d_2} - \frac{a_1 a_2}{\sum_{\beta_1 \beta_2}^{**}} \frac{a_1 a_2}{\sum_{\beta_1 \beta_2}^{**}} \eta_{\beta_1 \beta_2}^{e_1 e_2, f_1 f_2}(a_1 a_2^*) F_{\beta_1 \beta_2}^{e_1 e_2, f_1 f_2},$$

where $\sum_{\beta_1 \beta_2}^* \frac{a_1 a_2}{w_1 w_2}$ and $\sum_{\beta_1 \beta_2}^{**} \frac{a_1 a_2}{s_1 s_2}$ are the summations over all the values of $w_1 w_2$ and $s_1 s_2$ such that (I) if $\beta_1 \beta_2 = 00$ and (1) $a_1 a_2 = 00$, then $w_1 w_2 = 00$ and $s_1 s_2$ vanishes, (2) $a_1 a_2 = 10$, then $w_1 w_2 = 00, 10$ and $s_1 s_2 = 00$, (3) $a_1 a_2 = 01$, then $w_1 w_2 = 00, 10, 01$ and $s_1 s_2 = 00, 10$, (4) $a_1 a_2 = 20$, then $w_1 w_2 = 00, 10, 01, 20$ and $s_1 s_2 = 00, 10, 01$, (5) $a_1 a_2 = 02$, then $w_1 w_2 = 00, 10, 01, 20, 02$ and $s_1 s_2 = 00, 10, 01, 20$, and (6) $a_1 a_2 = 11$, then $w_1 w_2 = 00, 10, 01, 20, 02, 11$ and $s_1 s_2 = 00, 10, 01, 20, 02$, (II) if $\beta_1 \beta_2 = 10$ and (1) $a_1 a_2 = 10$, then $w_1 w_2 = 10$ and $s_1 s_2$ vanishes, (2) $a_1 a_2 = 20$ (if $m_1 \geq 3$), then $w_1 w_2 = 10, 20$ and $s_1 s_2 = 10$, and (3) $a_1 a_2 = 11$, then $w_1 w_2 = 10, 20$ (if $m_1 \geq 3$), 11 and $s_1 s_2 = 10, 20$ (if $m_1 \geq 3$), (III) if $\beta_1 \beta_2 = 01$ and (1) $a_1 a_2 = 01$, then $w_1 w_2 = 01$ and $s_1 s_2$ vanishes, (2) $a_1 a_2 = 02$ (if $m_2 \geq 3$), then $w_1 w_2 = 01, 02$ and $s_1 s_2 = 01$, and (3) $a_1 a_2 = 11$, then $w_1 w_2 = 01, 02$ (if $m_2 \geq 3$), 11 and $s_1 s_2 = 01, 02$ (if $m_2 \geq 3$), (IV) if $\beta_1 \beta_2 = 20$ ($m_1 \geq 4$) and $a_1 a_2 = 20$, then $w_1 w_2 = 20$ and $s_1 s_2$ vanishes, (V) if $\beta_1 \beta_2 = 02$ ($m_2 \geq 4$) and $a_1 a_2 = 02$, then $w_1 w_2 = 02$ and $s_1 s_2$ vanishes, and (VII) if $\beta_1 \beta_2 = a_1 a_2 = 11$, then $w_1 w_2 = 11$ and $s_1 s_2$ vanishes, respectively. Then the

following can be proved easily (see [8]):

Lemma 4.2. (i) The $P_{\beta_1\beta_2}^{a_1a_2}$'s are symmetric, mutually orthogonal and idempotent matrices.

$$(ii) \text{rank}(P_{\beta_1\beta_2}^{a_1a_2}) = \phi_{\beta_1\beta_2}.$$

First we consider a $2^{m_1+m_2}$ -PBFF design which is a PB-array with Condition (A). If $N \geq \nu(m_1m_2)$, then there may exist a design of resolution V. However if $N = \nu(m_1m_2)$, there is no d.f. due to error. Thus we consider the case in which $\{3(m_1+m_2)+m_1^2+m_2^2\}/2 (= \nu^A(m_1m_2))$, say $< N \leq \nu(m_1m_2)$. Let $P_0^A = I_N - \sum_{\beta_1\beta_2}^A \sum_{a_1a_2} P_{\beta_1\beta_2}^{a_1a_2}$, where the summation $\sum_{\beta_1\beta_2}^A$ is extended over all the values of $\beta_1\beta_2$ such that $\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$). Then it follows from Lemma 4.2 that $(P_0^A)^2 = P_0^A$, $P_0^A P_{\beta_1\beta_2}^{a_1a_2} = P_{\beta_1\beta_2}^{a_1a_2} P_0^A = 0_{N \times N}$ and $\text{rank}(P_0^A) = N - \nu^A(m_1m_2)$, where $0_{p \times q}$ denotes the $p \times q$ matrix with all zero. Let

$$\mathcal{R}_{\beta_1\beta_2}^{a_1a_2} = \mathcal{R}_{R(a_1a_2^*; \beta_1\beta_2) \perp (R(a_1a_2; \beta_1\beta_2))},$$

where

$$R(a_1a_2; \beta_1\beta_2) = [E_{\beta_1\beta_2} A^{\#(\beta_1\beta_2, \beta_1\beta_2)}; \dots; E_{a_1a_2} A^{\#(a_1a_2, a_1a_2)}] \text{ for } \beta_1\beta_2 = 00, 10, 01,$$

$$R(20; 20) = \begin{cases} 0_N & \text{if } m_1 = 2, 3, \\ [E_{20} A^{\#(20, 20)}] & \text{if } m_1 \geq 4, \end{cases}$$

$$R(02; 02) = \begin{cases} 0_N & \text{if } m_2 = 2, 3, \\ [E_{02} A^{\#(02, 02)}] & \text{if } m_2 \geq 4 \end{cases}$$

and $R(a_1a_2^*; \beta_1\beta_2)$'s are the matrices which are obtained by deleting $E_{a_1a_2} A^{\#(a_1a_2, a_1a_2)}$ from $R(a_1a_2; \beta_1\beta_2)$, and $R(\beta_1\beta_2^*; \beta_1\beta_2) = 0_N$.

Here $\mathcal{R}_{A \perp}(B)$ is the orthocomplement subspace of $\mathcal{R}(A)$ relative to $\mathcal{R}(B)$ for the case $\mathcal{R}(A) \subset \mathcal{R}(B)$, where $\mathcal{R}(A)$ denotes the linear subspace spanned by the column vectors of a matrix A . Then Lemma

4.2 and the properties of P_e^A yield the following:

Theorem 4.1. *Let T be a $2^{m_1+m_2}$ -PBFF design which is derived from a PB-array with Condition (A). Then we have*

$$\mathcal{R}^N = \begin{cases} \mathcal{R}_{00} \oplus \mathcal{R}_{10} \oplus \mathcal{R}_{01} \oplus \mathcal{R}_e^A & \text{if } m_1, m_2 = 2, 3, \\ \mathcal{R}_{00} \oplus \mathcal{R}_{10} \oplus \mathcal{R}_{01} \oplus \mathcal{R}_{20} \oplus \mathcal{R}_e^A & \text{if } m_1 \geq 4, m_2 = 2, 3, \\ \mathcal{R}_{00} \oplus \mathcal{R}_{10} \oplus \mathcal{R}_{01} \oplus \mathcal{R}_{02} \oplus \mathcal{R}_e^A & \text{if } m_1 = 2, 3, m_2 \geq 4, \\ \mathcal{R}_{00} \oplus \mathcal{R}_{10} \oplus \mathcal{R}_{01} \oplus \mathcal{R}_{20} \oplus \mathcal{R}_{02} \oplus \mathcal{R}_e^A & \text{if } m_1, m_2 \geq 4, \end{cases}$$

where \mathcal{R}^N is an N -dimensional vector space, \oplus denotes the direct sum, $\mathcal{R}_e^A = R_{E_T^\perp}$, which is the orthocomplement subspace of $\mathcal{R}(E_T)$ relative to \mathcal{R}^N , and

$$\mathcal{R}_{00} = \mathcal{R}_{00}^{00} \oplus \mathcal{R}_{00}^{10} \oplus \mathcal{R}_{00}^{01} \oplus \mathcal{R}_{00}^{20} \oplus \mathcal{R}_{00}^{02} \oplus \mathcal{R}_{00}^{11},$$

$$\mathcal{R}_{10} = \begin{cases} \mathcal{R}_{10}^{10} \oplus \mathcal{R}_{10}^{11} & \text{if } m_1 = 2, \\ \mathcal{R}_{10}^{10} \oplus \mathcal{R}_{10}^{20} \oplus \mathcal{R}_{10}^{11} & \text{if } m_1 \geq 3, \end{cases}$$

$$\mathcal{R}_{01} = \begin{cases} \mathcal{R}_{01}^{01} \oplus \mathcal{R}_{01}^{11} & \text{if } m_2 = 2, \\ \mathcal{R}_{01}^{01} \oplus \mathcal{R}_{01}^{02} \oplus \mathcal{R}_{01}^{11} & \text{if } m_2 \geq 3 \end{cases}$$

and

$$\mathcal{R}_{\beta_1\beta_2} = \mathcal{R}(R(\beta_1\beta_2; \beta_1\beta_2)) \text{ for } \beta_1\beta_2 = 20 \text{ (if } m_1 \geq 4), 02 \text{ (if } m_2 \geq 4).$$

Next consider a $2^{m_1+m_2}$ -PBFF design being a PB-array with Condition (B), and $\{3(m_1+2m_2)+m_1^2\}/2 (= \nu^B(m_1m_2))$, say $\langle N \leq \nu^A(m_1m_2) \rangle$, where $m_2 \geq 4$. Let $P_e^B = I_N - \sum_{\beta_1\beta_2}^B a_{\beta_1\beta_2} P_{\beta_1\beta_2}^{a_1a_2}$, where $\sum_{\beta_1\beta_2}^B$ is the summation over all the values of $\beta_1\beta_2$ such that $\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$). Then $(P_e^B)^2 = P_e^B$, $P_e^B P_{\beta_1\beta_2}^{a_1a_2} = P_{\beta_1\beta_2}^{a_1a_2} P_e^B = 0_{N \times N}$ and $\text{rank}(P_e^B) = N - \nu^B(m_1m_2)$.

Theorem 4.2. *For a $2^{m_1+m_2}$ -PBFF design T , which is a PB-array with Condition (B), we have*

$$\mathcal{R}^N = \begin{cases} \mathcal{R}_{00} \oplus \mathcal{R}_{10} \oplus \mathcal{R}_{01} \oplus \mathcal{R}_e^B & \text{if } m_1 = 2, 3, \\ \mathcal{R}_{00} \oplus \mathcal{R}_{10} \oplus \mathcal{R}_{01} \oplus \mathcal{R}_{20} \oplus \mathcal{R}_e^B & \text{if } m_1 \geq 4, \end{cases}$$

where $\mathcal{R}_{\beta_1\beta_2}$'s for $\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$) are given in Theorem

4.1, and $\mathcal{R}_e^B = \mathcal{R}_{E_T^\perp}$.

Finally consider a $2^{m_1+m_2}$ -PBFF design which is derived from a PB-array with Condition (C), where $3(m_1+m_2)$ ($=\nu^C(m_1m_2)$, say) $< N \leq \nu^B(m_1m_2)$ and $m_k \geq 4$. Let $P_e^C = I_N - \sum_{\beta_1\beta_2}^C \sum_{a_1a_2} P_{\beta_1\beta_2}^{a_1a_2}$, where $\sum_{\beta_1\beta_2}^C$ is the summation over all the values of $\beta_1\beta_2$ such that $\beta_1\beta_2 = 00, 10, 01$. Then $(P_e^C)^2 = P_e^C$, $P_e^C P_{\beta_1\beta_2}^{a_1a_2} = P_{\beta_1\beta_2}^{a_1a_2} P_e^C = 0_{N \times N}$ and $\text{rank}(P_e^C) = N - \nu^C(m_1m_2)$.

Theorem 4.3. *Let T be a $2^{m_1+m_2}$ -PBFF design which is a PB-array with Condition (C). Then*

$$\mathcal{R}^N = \mathcal{R}_{00} \oplus \mathcal{R}_{10} \oplus \mathcal{R}_{01} \oplus \mathcal{R}_e^C,$$

where $\mathcal{R}_{\beta_1\beta_2}$'s ($\beta_1\beta_2 = 00, 10, 01$) are the same as in Theorem 4.1, and $\mathcal{R}_e^C = \mathcal{R}_{E_T^\perp}$.

5. ANOVA and hypothesis testing

We first consider the ANOVA and the hypothesis testing of $2^{m_1+m_2}$ -PBFF designs of resolution IV satisfying Condition (A). Let $S_{\beta_1\beta_2}^{a_1a_2} = y_T' P_{\beta_1\beta_2}^{a_1a_2} y_T$ and $S_e^A = y_T' P_e^A y_T$. Then by Theorem 4.1, we have the following:

Theorem 5.1. *Let T be a $2^{m_1+m_2}$ -PBFF design which is a PB-array with Condition (A) and $\nu^A(m_1m_2) < N \leq \nu(m_1m_2)$. Then we have*

$$y_T' y_T = \sum_{\beta_1\beta_2}^A \sum_{a_1a_2} S_{\beta_1\beta_2}^{a_1a_2} + S_e^A.$$

Theorem 5.2. *For a design T of Theorem 5.1, an unbiased estimator of σ^2 is given by*

$$\hat{\sigma}^2 = S_e^A / \{N - \nu^A(m_1m_2)\}.$$

The noncentrality parameters, say, $\lambda_{\beta_1\beta_2}^{a_1a_2}/\sigma^2$, of the quadratic forms $y_T' P_{\beta_1\beta_2}^{a_1a_2} y_T / \sigma^2$ are defined by $\mathcal{E}[y_T'] P_{\beta_1\beta_2}^{a_1a_2} \mathcal{E}[y_T] / \sigma^2$, where

$\mathcal{E}[y]$ denotes the expected value of a random vector y . Let

$$c_{00}(p_1 p_2, q_1 q_2; a_1 a_2) = \begin{cases} \frac{a_1 a_2}{\sum_{c_1 c_2}^{(*)} 00} \frac{a_1 a_2}{\sum_{d_1 d_2}^{(*)} 00} \eta_{00}^{c_1 c_2, d_1 d_2} (a_1 a_2) \kappa_{00}^{p_1 p_2, c_1 c_2, d_1 d_2, q_1 q_2} \\ - \frac{a_1 a_2}{\sum_{e_1 e_2}^{(**)} 00} \frac{a_1 a_2}{\sum_{f_1 f_2}^{(**)} 00} \eta_{00}^{e_1 e_2, f_1 f_2} (a_1 a_2^*) \kappa_{00}^{p_1 p_2, e_1 e_2, f_1 f_2, q_1 q_2} \\ \text{if (1) } p_1 p_2 = q_1 q_2 = 11 \text{ for } a_1 a_2 = 11, \\ \text{(2) } p_1 p_2, q_1 q_2 = 02, 11 \text{ for } a_1 a_2 = 02, \\ \text{(3) } p_1 p_2, q_1 q_2 = 20, 02, 11 \text{ for } a_1 a_2 = 20, \\ \text{(4) } p_1 p_2, q_1 q_2 = 01, 20, 02, 11 \text{ for } a_1 a_2 = 01, \\ \text{(5) } p_1 p_2, q_1 q_2 = 10, 01, 20, 02, 11 \text{ for } a_1 a_2 = 10, \\ \text{(6) } p_1 p_2, q_1 q_2 = 00, 10, 01, 20, 02, 11 \text{ for } a_1 a_2 = 00, \\ 0 \text{ otherwise,} \end{cases}$$

where $\frac{a_1 a_2}{\sum_{w_1 w_2}^{(*)} 00}$ and $\frac{a_1 a_2}{\sum_{s_1 s_2}^{(**)} 00}$ are extended over all the values of $w_1 w_2$

and $s_1 s_2$ such that if $a_1 a_2 = 11$, then $w_1 w_2 = 11$ and $s_1 s_2$ vanishes, if $a_1 a_2 = 02$, then $w_1 w_2 = 02, 11$ and $s_1 s_2 = 11$, if $a_1 a_2 = 20$, then $w_1 w_2 = 20, 02, 11$ and $s_1 s_2 = 02, 11$, if $a_1 a_2 = 01$, then $w_1 w_2 = 01, 20, 02, 11$ and $s_1 s_2 = 20, 02, 11$, if $a_1 a_2 = 10$, then $w_1 w_2 = 10, 01, 20, 02, 11$ and $s_1 s_2 = 01, 20, 02, 11$, and if $a_1 a_2 = 00$, then $w_1 w_2 = 00, 10, 01, 20, 02, 11$ and $s_1 s_2 = 10, 01, 20, 02, 11$, respectively. Let

$$c_{10}(p_1 p_2, q_1 q_2; a_1 a_2) = \begin{cases} \frac{a_1 a_2}{\sum_{c_1 c_2}^{(*)} 10} \frac{a_1 a_2}{\sum_{d_1 d_2}^{(*)} 10} \eta_{10}^{c_1 c_2, d_1 d_2} (a_1 a_2) \kappa_{10}^{p_1 p_2, c_1 c_2, d_1 d_2, q_1 q_2} \\ - \frac{a_1 a_2}{\sum_{e_1 e_2}^{(**)} 10} \frac{a_1 a_2}{\sum_{f_1 f_2}^{(**)} 10} \eta_{10}^{e_1 e_2, f_1 f_2} (a_1 a_2^*) \kappa_{10}^{p_1 p_2, e_1 e_2, f_1 f_2, q_1 q_2} \\ \text{if (1) } p_1 p_2 = q_1 q_2 = 11 \text{ for } a_1 a_2 = 11, \\ \text{(2) } p_1 p_2, q_1 q_2 = 20 (m_1 \geq 3), 11 \text{ for } a_1 a_2 = 20, \\ \text{(3) } p_1 p_2, q_1 q_2 = 10, 20 (m_1 \geq 3), 11 \text{ for } a_1 a_2 = 10, \\ 0 \text{ otherwise,} \end{cases}$$

where $\frac{a_1 a_2}{\sum_{w_1 w_2}^{(*)} 10}$ and $\frac{a_1 a_2}{\sum_{s_1 s_2}^{(**)} 10}$ are the summations over all the values

of $w_1 w_2$ and $s_1 s_2$ such that $w_1 w_2 = 11$ and $s_1 s_2$ vanishes if $a_1 a_2 = 11$,

$w_1w_2=20$ (if $m_1 \geq 3$), 11 and $s_1s_2=11$ if $a_1a_2=20$, and $w_1w_2=10, 20$ (if $m_1 \geq 3$), 11 and $s_1s_2=20$ (if $m_1 \geq 3$), 11 if $a_1a_2=10$, respectively. Let

$$c_{01}(p_1p_2, q_1q_2; a_1a_2) = \begin{cases} \frac{a_1a_2}{\sum_{c_1c_2} \binom{a_1a_2}{(*)}_{01}} \frac{a_1a_2}{\sum_{d_1d_2} \binom{a_1a_2}{(*)}_{01}} \eta_{01}^{c_1c_2, d_1d_2} (a_1a_2) \kappa_{01}^{p_1p_2, c_1c_2, d_1d_2, q_1q_2} \\ - \frac{a_1a_2}{\sum_{e_1e_2} \binom{a_1a_2}{(**)_{01}} \sum_{f_1f_2} \binom{a_1a_2}{(**)_{01}} \eta_{01}^{e_1e_2, f_1f_2} (a_1a_2^*) \kappa_{01}^{p_1p_2, e_1e_2, f_1f_2, q_1q_2} \\ \text{if (1) } p_1p_2=q_1q_2=11 \quad \text{for } a_1a_2=11, \\ \text{(2) } p_1p_2, q_1q_2=02 \text{ (} m_2 \geq 3 \text{), 11} \quad \text{for } a_1a_2=02, \\ \text{(3) } p_1p_2, q_1q_2=01, 02 \text{ (} m_2 \geq 3 \text{), 11} \quad \text{for } a_1a_2=01, \\ 0 \quad \text{otherwise,} \end{cases}$$

where $\sum_{w_1w_2} \binom{a_1a_2}{(*)}_{01}$ and $\sum_{s_1s_2} \binom{a_1a_2}{(**)_{01}}$ are extended over all the values of w_1w_2

and s_1s_2 such that $w_1w_2=11$ and s_1s_2 vanishes if $a_1a_2=11$, $w_1w_2=02$ (if $m_2 \geq 3$), 11 and $s_1s_2=11$ if $a_1a_2=02$, and $w_1w_2=01, 02$ (if $m_2 \geq 3$), 11 and $s_1s_2=02$ (if $m_2 \geq 3$), 11 if $a_1a_2=01$, respectively. Further let

$$c_{20}(p_1p_2, q_1q_2; a_1a_2) = \begin{cases} \kappa_{20}^{20, 20} & \text{if } p_1p_2=q_1q_2=a_1a_2=20, \\ 0 & \text{otherwise,} \end{cases}$$

if $\det(K_{20}) \neq 0$ and $m_1 \geq 4$, and

$$c_{02}(p_1p_2, q_1q_2; a_1a_2) = \begin{cases} \kappa_{02}^{02, 02} & \text{if } p_1p_2=q_1q_2=a_1a_2=02, \\ 0 & \text{otherwise,} \end{cases}$$

if $\det(K_{02}) \neq 0$ and $m_2 \geq 4$. Then the following yields:

Theorem 5.3. Let T be a design of Theorem 5.1, then the non-centrality parameters of the quadratic forms $y_T' P_{\beta_1\beta_2}^{a_1a_2} y_T / \sigma^2$ for $\beta_1\beta_2=00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$) are

$$\lambda_{\beta_1\beta_2}^{a_1a_2} / \sigma^2 = \sum_{p_1p_2} \sum_{q_1q_2} \{ c_{\beta_1\beta_2}(p_1p_2, q_1q_2; a_1a_2) / \sigma^2 \} \times \theta_{p_1p_2}' A_{\beta_1\beta_2}^{\#(p_1p_2, q_1q_2)} \theta_{q_1q_2}.$$

Let $H_{\beta_1\beta_2}^{a_1a_2}$ be the hypotheses such that $A_{\beta_1\beta_2}^{\#(a_1a_2, a_1a_2)} \theta_{a_1a_2} =$

$0_{n(a_1a_2)}$ (if they exist). We are first interested in testing the

hypotheses $H_{\beta_1\beta_2}^{11}$ against $K_{\beta_1\beta_2}^{11}$ ($\beta_1\beta_2=00,10,01$), $H_{\beta_1\beta_2}^{20}$ against $K_{\beta_1\beta_2}^{20}$ (if $m_1 \geq 4$), and $H_{\beta_1\beta_2}^{02}$ against $K_{\beta_1\beta_2}^{02}$ (if $m_2 \geq 4$), where $K_{\beta_1\beta_2}^{a_1a_2}$'s are the hypotheses that $A_{\beta_1\beta_2}^{\#(a_1a_2, a_1a_2)} \Theta_{a_1a_2} \neq 0_{n(a_1a_2)}$. Next, if $H_{\beta_1\beta_2}^{11}$ (or $H_{\beta_1\beta_2}^{01}$) is accepted, we then consider the testing hypothesis $H_{\beta_1\beta_2}^{20}$ (if $m_1 \geq 3$) or $H_{\beta_1\beta_2}^{10}$ (if $m_1 = 2$) against $H_{\beta_1\beta_2}^{11}$ (or $H_{\beta_1\beta_2}^{01}$ (if $m_2 \geq 3$) or $H_{\beta_1\beta_2}^{01}$ (if $m_2 = 2$) against $H_{\beta_1\beta_2}^{11}$). If $H_{\beta_1\beta_2}^{20}$ is accepted, then we consider $H_{\beta_1\beta_2}^{02}$ against $H_{\beta_1\beta_2}^{11}$. Third, if $H_{\beta_1\beta_2}^{20}$ (if $m_1 \geq 3$) (or $H_{\beta_1\beta_2}^{02}$ (if $m_2 \geq 3$)) is accepted, then consider $H_{\beta_1\beta_2}^{10}$ against $H_{\beta_1\beta_2}^{20}$ (or $H_{\beta_1\beta_2}^{01}$ against $H_{\beta_1\beta_2}^{02}$), and if $H_{\beta_1\beta_2}^{02}$ is accepted, consider $H_{\beta_1\beta_2}^{20}$ against $H_{\beta_1\beta_2}^{02}$. If $H_{\beta_1\beta_2}^{20}$ is accepted, consider $H_{\beta_1\beta_2}^{01}$ against $H_{\beta_1\beta_2}^{20}$, and lastly if $H_{\beta_1\beta_2}^{01}$ is accepted, then consider $H_{\beta_1\beta_2}^{10}$ against $H_{\beta_1\beta_2}^{01}$. This method is the so-called nested test procedure (e.g., [2]). Notice that Theorem 5.3 implies that $\bigcap_{b_1b_2}^{a_1a_2} H_{\beta_1\beta_2}^{b_1b_2}$ is accepted if and only if $\lambda_{\beta_1\beta_2}^{a_1a_2} = 0$, where $\bigcap_{b_1b_2}^{a_1a_2} H_{\beta_1\beta_2}^{b_1b_2}$ denotes the intersection of $H_{\beta_1\beta_2}^{b_1b_2}$'s such that the running indices b_1b_2 have the same values as w_1w_2 of $\sum_{w_1w_2}^{a_1a_2} \binom{x}{w_1w_2} \beta_1\beta_2$ for $\beta_1\beta_2=00,10,01$, and as $\beta_1\beta_2$ for $\beta_1\beta_2=20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$). The test statistics for the nested method are given by

(1) for $\beta_1\beta_2=00$,

$$\frac{S_{000}^{11}/\phi_{00}}{S_e^A/\{N-\nu^A(m_1m_2)\}} (=F_{000}^{A11}, \text{ say}), \quad (5.1)$$

$$\frac{S_{000}^{02}/\phi_{00}}{\{S_e^A+S_{000}^{11}\}/\{N-\nu^A(m_1m_2)+\phi_{00}\}} (=F_{000}^{A02}, \text{ say}), \quad (5.2)$$

$$\frac{S_{000}^{20}/\phi_{00}}{\{S_e^A+S_{000}^{11}+S_{000}^{02}\}/\{N-\nu^A(m_1m_2)+2\phi_{00}\}} (=F_{000}^{A20}, \text{ say}), \quad (5.3)$$

$$\frac{S_{000}^{01}/\phi_{00}}{\{S_e^A+S_{000}^{11}+S_{000}^{02}+S_{000}^{20}\}/\{N-\nu^A(m_1m_2)+3\phi_{00}\}} (=F_{000}^{A01}, \text{ say}) \quad (5.4)$$

and

$$\frac{S_{00}^{10}/\phi_{00}}{\{S_e^A + S_{00}^{11} + S_{00}^{02} + S_{00}^{20} + S_{00}^{01}\}/\{N - \nu^A(m_1 m_2) + 4\phi_{00}\}} (=F_{00}^{A10}, \text{ say}), (5.5)$$

(ii) for $\beta_1 \beta_2 = 10$,

$$\frac{S_{10}^{11}/\phi_{10}}{S_e^A/\{N - \nu^A(m_1 m_2)\}} (=F_{10}^{A11}, \text{ say}), (5.6)$$

$$\frac{S_{10}^{20}/\phi_{10}}{\{S_e^A + S_{10}^{11}\}/\{N - \nu^A(m_1 m_2) + \phi_{10}\}} (=F_{10}^{A20}, \text{ say}) \text{ (if } m_1 \geq 3) (5.7a)$$

$$\text{(or } \frac{S_{10}^{10}/\phi_{10}}{\{S_e^A + S_{10}^{11}\}/\{N - \nu^A(m_1 m_2) + \phi_{10}\}} (=F_{10}^{A10}, \text{ say}) \text{ (if } m_1 = 2)) (5.7b)$$

and

$$\frac{S_{10}^{10}/\phi_{10}}{\{S_e^A + S_{10}^{11} + S_{10}^{20}\}/\{N - \nu^A(m_1 m_2) + 2\phi_{10}\}} (=F_{10}^{A10}, \text{ say}) \text{ (if } m_1 \geq 3) (5.8)$$

(iii) for $\beta_1 \beta_2 = 01$,

$$\frac{S_{01}^{11}/\phi_{01}}{S_e^A/\{N - \nu^A(m_1 m_2)\}} (=F_{01}^{A11}, \text{ say}), (5.9)$$

$$\frac{S_{01}^{02}/\phi_{01}}{\{S_e^A + S_{01}^{11}\}/\{N - \nu^A(m_1 m_2) + \phi_{01}\}} (=F_{01}^{A02}, \text{ say}) \text{ (if } m_2 \geq 3) (5.10)$$

$$\text{(or } \frac{S_{01}^{01}/\phi_{01}}{\{S_e^A + S_{01}^{11}\}/\{N - \nu^A(m_1 m_2) + \phi_{01}\}} (=F_{01}^{A01}, \text{ say}) \text{ (if } m_2 = 2))$$

and

$$\frac{S_{01}^{01}/\phi_{01}}{\{S_e^A + S_{01}^{11} + S_{01}^{02}\}/\{N - \nu^A(m_1 m_2) + 2\phi_{01}\}} (=F_{01}^{A01}, \text{ say}) \text{ (if } m_2 \geq 3) (5.11)$$

(iv) for $\beta_1 \beta_2 = 20$ and $m_1 \geq 4$,

$$\frac{S_{20}^{20}/\phi_{20}}{S_e^A/\{N - \nu^A(m_1 m_2)\}} (=F_{20}^{A20}, \text{ say}) (5.12)$$

and (v) for $\beta_1 \beta_2 = 02$ and $m_2 \geq 4$,

$$\frac{S_{02}^{02}/\phi_{02}}{S_e^A/\{N - \nu^A(m_1 m_2)\}} (=F_{02}^{A02}, \text{ say}).$$

All of them have F distributions, and the nesting procedure is continued until a significant test is obtained for each $\beta_1 \beta_2$.

Note that $F_{\beta_1 \beta_2}^{A a_1 a_2}$'s are central or noncentral F distributions with

$\phi_{\beta_1\beta_2}$ and $\{N - \nu^A(m_1m_2)\} + \tau^A(a_1a_2; \beta_1\beta_2)\phi_{\beta_1\beta_2}$ d.f., and noncentrality parameters $\lambda_{\beta_1\beta_2}^{a_1a_2}/\sigma^2$ depending on which $\bigcap_{b_1b_2}^{a_1a_2} H_{\beta_1\beta_2}^{b_1b_2}$ are true, where $\tau^A(a_1a_2; \beta_1\beta_2)$'s are some integers as above.

Next consider the ANOVA and the hypothesis testing of $2^{m_1+m_2}$ -PBFF designs of resolution IV which satisfy Condition (B).

Theorem 5.4. *Let T be a $2^{m_1+m_2}$ -PBFF design which is a PB-array with Condition (B) and $\nu^B(m_1m_2) < N \leq \nu^A(m_1m_2)$. Then*

$$y_T' y_T = \sum_{\beta_1\beta_2}^B \sum_{a_1a_2} S_{\beta_1\beta_2}^{a_1a_2} + S_e^B,$$

where $S_e^B = y_T' P_e^B y_T$.

Theorem 5.5. *For a design T of Theorem 5.4, an unbiased estimator of σ^2 is*

$$\hat{\sigma}^2 = S_e^B / \{N - \nu^B(m_1m_2)\}.$$

Theorem 5.6. *Let T be a design of Theorem 5.4. Then the noncentrality parameters of the quadratic forms $y_T' P_{\beta_1\beta_2}^{a_1a_2} y_T / \sigma^2$ for $\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$) are given by*

$$\lambda_{\beta_1\beta_2}^{a_1a_2} / \sigma^2 = \sum_{p_1p_2} \sum_{q_1q_2} \{c_{\beta_1\beta_2}(p_1p_2, q_1q_2; a_1a_2) / \sigma^2\} \times \theta_{p_1p_2}^{A\#(p_1p_2, q_1q_2)} \theta_{q_1q_2}.$$

We now consider the hypotheses $H_{\beta_1\beta_2}^{11}$ against $K_{\beta_1\beta_2}^{11}$ for $\beta_1\beta_2 = 00, 10, 01$, $H_{\beta_1\beta_2}^{20}$ against $K_{\beta_1\beta_2}^{20}$ (if $m_1 \geq 4$). Next if $H_{\beta_1\beta_2}^{10}$ (or $H_{\beta_1\beta_2}^{01}$) is accepted, consider the testing hypothesis $H_{\beta_1\beta_2}^{20}$ (if $m_1 \geq 3$) or $H_{\beta_1\beta_2}^{10}$ (if $m_1 = 2$) against $H_{\beta_1\beta_2}^{11}$ (or $H_{\beta_1\beta_2}^{01}$ against $H_{\beta_1\beta_2}^{11}$). If $H_{\beta_1\beta_2}^{10}$ is accepted, then consider $H_{\beta_1\beta_2}^{00}$ against $H_{\beta_1\beta_2}^{10}$. Third, if $H_{\beta_1\beta_2}^{20}$ (if $m_1 \geq 3$) (or $H_{\beta_1\beta_2}^{01}$) is accepted, then consider $H_{\beta_1\beta_2}^{10}$ against $H_{\beta_1\beta_2}^{20}$ (or $H_{\beta_1\beta_2}^{01}$ against $H_{\beta_1\beta_2}^{20}$), and if $H_{\beta_1\beta_2}^{00}$ is accepted, consider $H_{\beta_1\beta_2}^{20}$ against $H_{\beta_1\beta_2}^{00}$. If $H_{\beta_1\beta_2}^{20}$ is accepted, consider $H_{\beta_1\beta_2}^{01}$ against $H_{\beta_1\beta_2}^{20}$, and lastly if $H_{\beta_1\beta_2}^{00}$ is accepted, then consider $H_{\beta_1\beta_2}^{10}$ against $H_{\beta_1\beta_2}^{00}$. Note that Theorem 5.6 means that

$a_1 a_2$
 $b_1 b_2$ $H_{\beta_1 \beta_2}^{b_1 b_2}$ is accepted if and only if $\lambda_{\beta_1 \beta_2}^{a_1 a_2} = 0$. The test statistics, say $F_{\beta_1 \beta_2}^{a_1 a_2}$, for the nested method are given by replacing S_e^A and $\nu^A(m_1 m_2)$ of (5.1) through (5.12) with S_e^B and $\nu^B(m_1 m_2)$, respectively. The $F_{\beta_1 \beta_2}^{a_1 a_2}$'s have F distributions similar to $F_{\beta_1 \beta_2}^{a_1 a_2}$'s.

We finally consider the ANOVA and the hypothesis testing of $2^{m_1+m_2}$ -PBFF designs satisfying Condition (C).

Theorem 5.7. Let T be a $2^{m_1+m_2}$ -PBFF design which is a PB-array with Condition (C) and $\nu^C(m_1 m_2) < N \leq \nu^B(m_1 m_2)$. Then we have

$$y' T y_T = \sum_{\beta_1 \beta_2}^C \sum_{a_1 a_2} S_{\beta_1 \beta_2}^{a_1 a_2} + S_e^C,$$

where $S_e^C = y' T P_e^C y_T$.

Theorem 5.8. Let T be a design of Theorem 5.7, then an unbiased estimator of σ^2 is given by

$$\hat{\sigma}^2 = S_e^C / \{N - \nu^C(m_1 m_2)\}.$$

Theorem 5.9. For a design T of Theorem 5.7, the noncentrality parameters of the quadratic forms $y' T P_{\beta_1 \beta_2}^{a_1 a_2} y_T / \sigma^2$ ($\beta_1 \beta_2 = 00, 10, 01$) are

$$\lambda_{\beta_1 \beta_2}^{a_1 a_2} / \sigma^2 = \sum_{p_1 p_2} \sum_{q_1 q_2} \{c_{\beta_1 \beta_2}(p_1 p_2, q_1 q_2; a_1 a_2) / \sigma^2\} \\ \times \theta_{p_1 p_2}^{a_1 a_2} A_{\beta_1 \beta_2}^{\#(p_1 p_2, q_1 q_2)} \theta_{q_1 q_2}.$$

Consider the testing hypotheses $H_{\beta_1 \beta_2}^{11}$ against $K_{\beta_1 \beta_2}^{11}$ for $\beta_1 \beta_2 = 00, 10, 01$. Next if H_{10}^{11} (or H_{01}^{11}) is accepted, then consider the testing hypothesis H_{10}^{20} against H_{10}^{11} (or H_{01}^{20} against H_{01}^{11}). If H_{10}^{20} is accepted, then consider H_{10}^{20} against H_{10}^{11} . Third if H_{10}^{20} (or H_{01}^{20}) is accepted, consider H_{10}^{20} against H_{10}^{20} (or H_{01}^{20} against H_{01}^{20}), and if H_{10}^{20} is accepted, consider H_{10}^{20} against H_{10}^{20} . If H_{10}^{20} is ac-

cepted, consider H_{00}^{01} against H_{00}^{20} , and lastly if H_{00}^{01} is accepted, consider H_{00}^{10} against H_{00}^{01} . Note that Theorem 5.9 implies that $a_1 a_2$
 $b_1 b_2$ $H_{\beta_1 \beta_2}^{b_1 b_2}$ is accepted if and only if $\lambda_{\beta_1 \beta_2}^{a_1 a_2} = 0$. The test statistics, say $F_{\beta_1 \beta_2}^{ca_1 a_2}$, for the nested method are given by replacing S_e^A and $\nu^A(m_1 m_2)$ of (5.1) through (5.11) with S_e^C and $\nu^C(m_1 m_2)$, respectively. The $F_{\beta_1 \beta_2}^{ca_1 a_2}$'s have F distributions similar to $F_{\beta_1 \beta_2}^{Aa_1 a_2}$'s.

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